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AN ITERATIVE METHOD OF SOLVING THE PROBLEM OF INTERACTION BETWEEN ELASTIC BODIES*

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A finite-dimensional analogue of the static three-dimensional contact problem for two elastic bodies is studied. The interacting bodies are approximated by half-spaces, taking Coulomb friction into account. It is assumed that the normal ealstic displacements are independent of the tangential stresses and that the boundary separating the slippage zones from the adhesion zones within the area of contact is reduced to a system of non-linear equations. An iterative process is proposed for solving this system. The convergence of the iterative sequence to a unique limit point independent of the initial approximation, is shown.

Let us consider the static, three-dimensional problem of contact between two elastic bodies, taking Coulomb friction into account, with partial slippage and adhesion taking place within the area of contact /1/. We shall approximate the bodies in contact by half-spaces. In a number of important practical cases a decomposition of the initial problem is possible such that the area of contact and the normal stresses can be found independently of the determination of tangential stresses and relative displacements /2, 3/. The classical formulation of the second of these problems /1/ will be given as follows. Assuming that the function $y(\xi, \eta)$ of normal stresses is known, it is required to find, within the area of contact, a vector function of tangential stresses $\mathbf{x}(\xi, \eta)$ and a vector function of the relative displacements $\mathbf{V}(\xi, \eta)$ satisfying the conditions

$$|\mathbf{x}| \leqslant \mu |y|, \quad \mathbf{V} \neq 0 \to \mathbf{x} = -\mu |y| |\mathbf{V}|^{-1}\mathbf{V}$$

Here $\mathbf{V} = B(\mathbf{x}) + \mathbf{F}$, $B(\mathbf{x})$ is a linear integral operator with a weak singularity /2, 3/, $\mathbf{F} = \mathbf{F}(\xi, \eta)$ is the rigid, tangential relative displacement of the bodies and μ is the coefficient of friction. We shall call the following system of relations the finite-dimensional problem of determining the tangential stresses and relative displacements (problem A):

$$|\mathbf{x}_i| \leqslant \mu |y_i|, \quad |\mathbf{V}_i| = 0, \quad i \in I_1 \subset I_0 \tag{1}$$

$$|\mathbf{V}_i| > 0, \quad \mathbf{x}_i = -\mu |y_i| \, |\mathbf{V}_i|^{-1}\mathbf{V}_i, \quad i \in I_2 \subset I_0 \tag{2}$$

Here

$$\mathbf{x}_i = (x_{2i-1}, x_{2i}), \ \mathbf{V}_i = (V_{2i-1}, V_{2i}); \ I_1 \cup I_2 = I_0 = \{1, \ldots, N\}, \ I_1 \cap I_2 = \varnothing;$$

$$V_k = \sum_{j=1}^{2N} b_{kj}x_j + F_k, \quad k = 1, \ldots, 2N$$

and the matrix $\|b_{kj}\|$ is symmetrical and positive definite /2, 3/.

In accordance with the meaning of the problem, $y_i \leqslant 0$ is the normal stress at the i-th point on the surface of one of the bodies in contact, $\mathbf{x}_i, \mathbf{V}_i, \mathbf{F}_i = (F_{2i-1}, F_{2i})$ are the vectors of tangential stress, the relative displacement of the surfaces of the bodies under load, and of the rigid relative displacement of the i-th point respectively, $\|b_{kj}\|$ is the matrix of the coefficients of the effect of the j-th component of the tangential stress vector at the $[(j+1)/2]$-th point on the k-th component of elastic tangential displacement vector at the point with number $[(k+1)/2]$, where $[\cdot]$ denotes the integral part of the number. The sets I_1 and I_2 consist of the indices of the points situated within the adhesion and slippage zones respectively, and must be determined when solving system (1), (2).

We shall assume that the problem of determining the normal stresses and the area of contact has been solved, i.e. that the numbers $y_i \ (i = 1, \ldots, N)$ are known. The formulation of this problem (we shall call it problem B) and one of the methods for solving it were given in /4/.

Let us turn our attention to problem A. Dividing the second relation of (1) and the first relation of (2) by $\beta_i = \max\{b_{2i-1, 2i-1}, b_{2i, 2i}\} + |b_{2i-1, 2i}| > 0$, we obtain the system

$$|\mathbf{x}_i| \leqslant z_i, \quad |\Delta_i| = 0, \quad i \in I_1 \subset I_0 \tag{3}$$

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$$|\Delta_i|>0,\ \mathbf{x}_i=-z_i|\Delta_i|^{-1}\Delta_i,\ i\in I_2\subset I_0 \qquad (4)$$
$$z_i=\mu|y_i|,\ \Delta_i=(\Delta_{2i-1},\Delta_{2i}),\ \Delta_k=\sum_{j=1}^{2N}a_{kj}x_j+f_k$$
$$a_{2i-1,j}=b_{2i-1,j}/\beta_i,\ a_{2i,j}=b_{2i,j}/\beta_i,\ f_{2i-1,j}=F_{2i-1,j}/\beta_i$$
$$f_{2i,j}=F_{2i,j}/\beta_i,\ i=1,\ldots,N;\ k,j=1,\ldots,2N$$

Let us consider a system of non-linear equations for the unknown vectors $\mathbf{x}_i$ $(i=1,\ldots,N)$

$$\mathbf{x}_i=(\mathbf{x}_i-\Delta_i)q_i \qquad (5)$$
$$q_i=q(|\mathbf{x}_i-\Delta_i|,z_i)=\begin{cases}1, & |\mathbf{x}_i-\Delta_i|\leqslant z_i\\ z_i|\mathbf{x}_i-\Delta_i|^{-1}, & |\mathbf{x}_i-\Delta_i|>z_i\end{cases}$$

Theorem 1. The system of Eqs.(5) is equivalent, for the vectors $\mathbf{x}_i$ $(i=1,\ldots,N)$, to the system of relations (3), (4).

Proof. Let $\mathbf{x}_i$ $(i=1,\ldots,N)$ satisfy system (5). For the values of i for which $|\mathbf{x}_i-\Delta_i|\leqslant z_i$, we have $q_i=1$. Therefore $|\Delta_i|=0$, $|\mathbf{x}_i|\leqslant z_i$, i.e. relations (3) hold. For the values of i for which $|\mathbf{x}_i-\Delta_i|>z_i$, we have $\mathbf{x}_i=(\mathbf{x}_i-\Delta_i)z_i|\mathbf{x}_i-\Delta_i|^{-1}$. This yields $|\mathbf{x}_i|=z_i$, and the first relation of (4) holds. If in addition $z_i=0$, then $|\mathbf{x}_i|=0$ and the second relation of (4) holds. If on the other hand $z_i>0$, then $\Delta_i=-\alpha_i\mathbf{x}_i$, where $\alpha_i=|\mathbf{x}_i-\Delta_i|(z_i)^{-1}-1>0$ and again the second relation of (4) holds. We will show in the same manner that if $\mathbf{x}_i$ $(i=1,\ldots,N)$ satisfies the relations (3), (4), then system (5) is also satisfied.

Let us carry out the following iterative process of solving system (5). Suppose, in the n-th approximation,

$$|\mathbf{x}_m{}^n|\leqslant z_m,\ m=1,\ldots,N \qquad (6)$$

We compute the components of the vector $\Delta_i{}^n$ and the quantity $q_i{}^n$:

$$\Delta_{2i-1}^{(n)}=\sum_{j=1}^{2N}a_{2i-1,j}x_j{}^n+f_{2i-1},\ \Delta_{2i}^n=\sum_{j=1}^{2N}a_{2i,j}x_j+f_{2i} \qquad (7)$$

$$q_i{}^n=\begin{cases}1, & R_i{}^n\leqslant z_i\\ z_i/R_i{}^n, & R_i{}^n>z_i;\quad R_i{}^n=|\mathbf{x}_i{}^n-\Delta_i{}^n|\end{cases} \qquad (8)$$

$$i=n+1-rN,\ r-\ \text{is an integer such that}\ 1\leqslant i\leqslant N \qquad (9)$$

We assume that

$$\mathbf{x}_m^{n+1}=\mathbf{x}_m{}^n+\delta_m{}^n \qquad (10)$$

where the vectors $\delta_m{}^n=(\delta_{2m-1}^n,\delta_{2m}^n)$ are defined as follows:

$$\delta_m{}^n=0\quad\text{when}\quad m\neq i \qquad (11)$$
$$\delta_m{}^n=-\mathbf{x}_m{}^n+(\mathbf{x}_m{}^n-\Delta_m{}^n)q_m{}^n\quad\text{when}\quad m=i$$

(the value of the index i is connected with the relation (9) by the number of the approximation n).

We shall show that $|\mathbf{x}_m^{n+1}|\leqslant z_m$. For $m\neq i$ this is obvious. Let $m=i$. If $|\mathbf{x}_i{}^n-\Delta_i{}^n|\leqslant z_i$, then $\mathbf{x}_i^{n+1}=\mathbf{x}_i{}^n-\Delta_i{}^n$, therefore $|\mathbf{x}_i^{n+1}|\leqslant z_i$. If $|\mathbf{x}_i{}^n-\Delta_i{}^n|>z_i$, then $\mathbf{x}_i^{n+1}=(\mathbf{x}_i{}^n-\Delta_i{}^n)z_i|\mathbf{x}_i{}^n-\Delta_i{}^n|^{-1}$, and hence $|\mathbf{x}_i^{n+1}|=z_i$. This implies that in order for inequality (6) to hold, it is only necessary to assume, in the zeroth approximation, that $|\mathbf{x}_m{}^0|\leqslant z_m$, $m=1,\ldots,N$.

Theorem 2. A unique solution of system (5) exists, which can be obtained as a limit of the sequence constructed according to the rule given by (7)–(11).

Proof. Let us consider the functional

$$l(\mathbf{x})=l(x_1,\ldots,x_{2N})=\sum_{k=1}^{2N}(x_k-X_k)\Big(\sum_{j=1}^{2N}b_{kj}x_j+F_k\Big)=\sum_{k=1}^{2N}\sum_{j=1}^{2N}b_{kj}(x_k-X_k)(x_j-X_j)$$

where $\mathbf{X}=(X_1,\ldots,X_{2N})$ is a solution of the system of equations

$$\sum_{j=1}^{2N}b_{kj}x_j+F_k=0,\quad k=1,\ldots,2N \qquad (12)$$

Since the matrix $\|b_{kj}\|$ is positive definite, $l(x_1,\ldots,x_{2N})\geqslant 0$ and a solution of system (12) exists and is unique.

The functional introduced is the same as that discussed in /4/. Carrying out transformations similar to those in /4/, we obtain

$$l(\mathbf{x}^{n+1})-l(\mathbf{x}^n)=2\beta_i(\delta_{2i-1}^n\Delta_{2i-1}^n+\delta_{2i}^n\Delta_{2i}^n)+ \qquad (13)$$
$$b_{2i-1,2i-1}(\delta_{2i-1}^n)^2+2b_{2i-1,2i}\delta_{2i-1}^n\delta_{2i}^n+b_{2i,2i}(\delta_{2i}^n)^2$$

It can be shown that

$$p_i^n = (\delta_{2i-1}^n)^2 + (\delta_{2i}^n)^2 + \delta_{2i-1}^n\Delta_{2i-1}^n + \delta_{2i}^n\Delta_{2i}^n \leqslant 0$$

Indeed,

$$p_i^n = |\mathbf{x}_i^n - \Delta_i^n|^2 (q_i^n - 1) q_i^n - |\mathbf{x}_i^n| \, |\mathbf{x}_i^n - \Delta_i^n| \cos\varphi_i^n (q_i^n - 1) \tag{14}$$

φ_i^n is the angle between the vectors $\mathbf{x}_i^n$ and $(\mathbf{x}_i^n - \Delta_i^n)$).

If $|\mathbf{x}_i^n - \Delta_i^n| \leqslant z_i$, then $q_i^n = 1$ and we have $p_i^n = 0$. Let $|\mathbf{x}_i^n - \Delta_i^n| > z_i$. Remembering that at every iteration step $|\mathbf{x}_i^n| \leqslant z_i$, we shall write

$$\frac{|\mathbf{x}_i^n| \cos\varphi_i^n}{|\mathbf{x}_i^n - \Delta_i^n|} \leqslant \frac{z_i}{|\mathbf{x}_i^n - \Delta_i^n|} = q_i^n$$

and this shows that the right-hand side of Eq.(14) is non-positive, which it was required to prove.

Returning to relation (13), we write the estimate

$$l(\mathbf{x}^{n+1}) - l(\mathbf{x}^n) \leqslant -C|\boldsymbol{\delta}^n|^2$$

$$C = \min_{1 \leqslant i \leqslant N} \beta_i > 0, \quad \boldsymbol{\delta}^n = (0, \ldots, \delta_{2i-1}^n, \delta_{2i}^n, \ldots, 0)$$

Since $\boldsymbol{\delta}^n = \mathbf{x}^{n+1} - \mathbf{x}^n$, therefore $l(\mathbf{x}^{n+1}) \leqslant l(\mathbf{x}^n) - C|\mathbf{x}^{n+1} - \mathbf{x}^n|^2$. It can be shown that the following relation holds for any integral value of $t > 0$:

$$0 \leqslant |\mathbf{x}^{n+t} - \mathbf{x}^n|^2 \leqslant tC^{-1}[l(\mathbf{x}^n) - l(\mathbf{x}^{n+t})]$$

therefore a limit $\mathbf{x}^n|_{n\to\infty} = \mathbf{x} = (x_1, \ldots, x_{2N})$ exists, which will represent the solution of system (5).

The solution is unique. Indeed, let two solutions exist $\mathbf{x}_i^* = (x_{2i-1}^*, x_{2i}^*)$, $\mathbf{x}_i^{**} = (x_{2i-1}^{**}, x_{2i}^{**})$, $i = 1, \ldots, N$. Consider the relation

$$S = \sum_{k=1}^{2N}\sum_{j=1}^{2N} b_{kj}(x_j^* - x_j^{**})(x_k^* - x_k^{**}) =$$
$$\sum_{k=1}^{2N}\Big(\sum_{j=1}^{2N} b_{kj}x_j^* - \sum_{j=1}^{2N} b_{kj}x_j^{**}\Big)(x_k^* - x_k^{**}) = \sum_{i=1}^{N}\beta_i S_i$$

Here

$$S_i = (\Delta_{2i-1}^* - \Delta_{2i}^{**})(x_{2i-1}^* - x_{2i-1}^{**}) + (\Delta_{2i}^* - \Delta_{2i}^{**})(x_{2i}^* - x_{2i}^{**}), \quad \Delta_k^* = \sum_{j=1}^{2N} a_{kj}x_j^* + f_k$$
$$\Delta_k^{**} = \sum_{j=1}^{2N} a_{kj}x_j^{**} + f_k, \quad k = 1, \ldots, 2N$$

The following combinations of the quantities $q_i^* = q(R_i^*, z_i)$, $q_i^{**} = q(R_i^{**}, z_i)$ are possible at various values of the index i calculated from the second relation of (5):

$$q_i^* = 1, \ q_i^{**} = 1; \ q_i^* = 1, \ q_i^{**} = z_i/R_i^{**}$$
$$q_i^* = z_i/R_i^*, \ q_i^{**} = 1; \ q_i^* = z_i/R_i^*, \ q_i^{**} = z_i/R_i^{**}$$
$$(R_i^* = |\mathbf{x}_i^* - \Delta_i^{**}|, \ R_i^{**} = |\mathbf{x}_i^{**} - \Delta_i^{**}|)$$

It can be shown that $S_i \leqslant 0$ for any of these combinations. But then we also have $S \leqslant 0$, which is possible when the matrix $\|b_{kj}\|$ is positive definite, only when $\mathbf{x}_i^* = \mathbf{x}_i^{**}$, $i = 1, \ldots, N$, which proves Theorem 2.

The iterative process proposed here, just like the process discussed in /4/, is an analogue of the Gauss relaxation method of solving systems of linear algebraic equations /5/. The algorithm (7)-(11) ensures the convergence of the iterative sequence $\mathbf{x}^n$ $(n = 0, 1, \ldots)$ irrespective of the choice of the initial approximation, since after the first cycle of N iterations conditions (6) hold.

As an example, we have solved the problem of contact between two cylinders with mutually orthogonal generatrices, made of the same material and loaded by normal and shearing forces, The rectangular region including the area of contact was covered by a 20 x 20 mesh, and the stresses sought were calculated at the centre of each elementary cell. To solve problem B, we used the process described in /4/: problem A was solved using algorithm (7)-(11). The following inequality was satisfied after 25 cycles:

$$|\mathbf{x}^{(n+1)N} - \mathbf{x}^{nN}| / |\mathbf{x}^{(n+1)N}| \leqslant 10^{-5}$$

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NEUTRAL LOADING IN THE ENDOCHRONIC MODEL OF THE THEORY OF PLASTICITY*

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The properties of a simple model of the endochronic theory of plasticity (ETP) for inactive deformation processes are studied. The theory uses "internal time" as the parameter of the deformation process, and no distinction is made between the loading and unloading which is essentially non-linear /1–4/.

A loading and deformation processes are described in a five-dimensional stress deviator space Σ_5 /5/. Restricting ourselves to processes not depending explicitly on time, we can write the defining equations of ETP for an initially isotropic material in the form

$$de = de_p + de_e,\ de_e = E^{-1}d\sigma,\ de_p = \sigma dz,\ dz = F(\sigma, e, e', \xi)\, d\xi, \quad d\xi = |de - \chi E^{-1}d\sigma|,\ 0 \leqslant \chi \leqslant 1 \tag{1}$$

Here e, e_p, e_e are the total, plastic and elastic deformation vectors, respectively, σ is the stress vector, z is the "internal time" parameter, E is the modulus of elasticity, F is the O (5)-invariant hardending function, /4/, ξ is the modified measure of the deformation, and χ is a parameter characterizing the contribution of the elastic component of the deformation towards the variation in intrinsic time; for any $a, a' = da/ds$, where $ds = |de|$ is the length of the arc of the deformation trajectory. We assume that the magnitude of the volumetric deformation ε does not affect the plasticity.

System (1) yields the following equation:

$$d\sigma = Ede - EF\sigma d\xi \tag{2}$$

In the simplest case F can be regarded as a constant of the material, in which case Eq. (2) will take the form ($\sigma_0 = 1/F$ is the yield point)

$$d\sigma = Ede - \alpha\sigma\, |de - \chi E^{-1}d\sigma|,\ \alpha = E/\sigma_0 \tag{3}$$

The above equation describes a material without hardening. For a material with linear isotropic hardening we must make the substitution $\sigma_0 \to \sigma_0(1 + k\xi)$ /1/ (in the case of translational hardening $\sigma \to \sigma - \mu e$, μ is the hardening modulus).

The ETP has no concept of a yield surface and there is no unloading condition; therefore Eqs.(2), (3) are assumed to hold for both the loading (active processes $\sigma de > 0$) and unloading (passive processes $\sigma de < 0$ /6/).

Let us write the condition of inactivity of the process: $\sigma de \leqslant 0$ (the equality corresponds to neutral loading). Substituting de into it from (2), we can rewrite this condition in the form

$$\sigma_u' + EF\sigma_u\xi' \leqslant 0,\ \sigma_u = |\sigma| \tag{4}$$

Let γ be the angle between σ and $d\sigma$, $a = |\sigma'|$. Then

$$\sigma_u' = a\cos\gamma \tag{5}$$

Let us transform the expression for ξ'. To do this, we substitute de from (2) into the expression for ξ' (1) and square the resulting expression. After cancelling like terms, we obtain a quadratic equation for ξ', whose root is

$$\xi' = \frac{1-\chi}{E} a \frac{F\sigma_u\cos\gamma + \sqrt{1 - F^2\sigma_u^2\sin^2\gamma}}{1 - F^2\sigma_u^2} \tag{6}$$

(the second root is rejected as extraneous). Substituting (6) into (4) we write, after reduction, the condition of inactivity of the process when an extra load $d\sigma$ is added to the

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